TWO-PHASE POTENTIAL FLOW

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NOMENCLATURE

$\mathbf{d\ell}, \quad \mathbf{ds}$	Elements of length, area
E	Exertia, $(\alpha_1\beta - 1)$
$\mathbf{f}$	Force per unit volume of a phase
$\mathbf{g}$	Body force per unit mass
$\mathbf{I}$	Unit dyadic
$\mathbf{j}$	Volumetric flux density
k	Kinetic energy density
m	Geurst's added mass coefficient
$\mathbf{m}$	Momentum density
$\mathbf{M}_1^d$	Added mass force
p	Mean pressure in a phase
P	Macroscopic pressure
$\mathbf{P}$	Combined momentum flux and stress tensor
R	Density ratio ρ_1/ρ_2
t	Time
$\mathbf{u}$	Microscopic velocity
$\mathbf{v}$	Average velocity
$\mathbf{w}$	Relative velocity $\mathbf{v}_1 - \mathbf{v}_2$
α	Volumetric fraction of a phase
β	"Resistivity," (1)
η	Macroscopic potential
$\boldsymbol{\pi}$	Specific momentum, (24), (25)
ρ	Density
Φ	Macroscopic potential

Subscripts

0	With particles at rest
1	Phase 1
2	Phase 2

INTRODUCTION

Potential flow theory for a single fluid has been established for many years. Although its limitations for describing real motions are well known, it does provide a self-consistent structure for analysis and often provides a reasonably accurate description of at least part of the flow field of fluids with low viscosity.

As an example of two-phase flow, one can imagine a suspension of particles in a fluid that obeys all the requirements of potential theory at the microscopic level. If the only forces acting on the particles are "conservative," it would appear that their motion might also reasonably be expected to be describable in terms of a suitable potential. Averaging of these potentials would lead to *macroscopic* potentials, true *properties* of the mixture, that should be related in some way to the average motion of the phases. Indeed, previous attempts to determine the inertial coupling terms in the *two-fluid model* have implicitly assumed potential flow at the microscopic level.

A complete two-phase theory of this type will be as *idealized* as was the classical theory of single-phase potential flow. However, it should be useful in the same sorts of ways, both as an approximation in many situations and as a *standard* that must at some level be consistent with other approaches, such as those that attempt to define closure relations for a set of averaged basic equations.

This paper describes some features of two recent approaches along these lines. The first is based on a set of *progressive examples* that can be analyzed using common techniques, such as conservation laws, and taken together appear to lead in the direction of a general theory, the tactic used in [1]. The second is based on *variational methods*, a classical approach to conservative mechanical systems that has a respectable history of application to single phase flows. The latter approach, exemplified by several recent papers by Geurst

[2-5], appears generally to be consistent with the former, at least in those cases for which it has been possible to obtain comparable results.

PROGRESSIVE EXAMPLES

The theory developed in [1] starts with a situation where the particles are at rest. Fluid flows past these particles as though a *porous medium*. At the microscopic level the equipotentials are not "smooth" but they do not differ much from the more gross macroscopic equipotential. Differences between the two levels of equipotential are smeared out over lengths comparable with the particle size. I do not have a rigorous proof, but I believe it is valid to treat these two equipotential surfaces as essentially identical for most purposes. This is not true of the other properties, such as pressure and velocity, that vary more at the microscopic than the macroscopic scale and must be averaged carefully.

Just as in electrical conduction past a matrix of non-conducting particles, the macroscopic fluid flux will be proportional to the macroscopic potential gradient, the "resistivity" being represented by a factor β that depends only on the particle geometry and the void fraction, as long as the arrangement is isotropic. We therefore have

$$\mathbf{j}_0 = -\frac{1}{\beta}\nabla\Phi \qquad (1)$$

Since β is unity for unimpeded flow, it will be greater than one when particles are present. A crude description of the situation could be to say that some of the fluid is "held up" or "entrained" by the particles so that only a fraction of the space is available for direct flow.

The average velocity of the fluid is the *relative velocity*,

$$\mathbf{w} = \frac{\mathbf{j}_0}{\alpha_1} = -\frac{1}{\alpha_1\beta}\nabla\Phi \qquad (2)$$

The kinetic energy of the fluid in unit volume confined between two equipotentials is, from a standard theorem of potential flow.

$$k_0 = -\frac{1}{2}\rho_1 \int_s \phi \mathbf{u} \cdot \mathbf{ds} = -\frac{1}{2}\rho_1 \nabla\Phi \cdot \mathbf{j}_0 = \frac{1}{2}\rho_1 \alpha_1^2 \beta w^2 \tag{3}$$

The kinetic energy per unit volume of phase 1 is

$$k_1 = \frac{k_0}{\alpha_1} = \frac{1}{2}\rho_1 w^2 (\alpha_1 \beta) \tag{4}$$

If we denote the fluid velocity at the microscopic level by $\mathbf{u}_1$, (2) and (4) are equivalent to

$$\mathbf{w} = <\mathbf{u}_1> \tag{5}$$

$$w^2 \alpha_1 \beta = <\mathbf{u}_1^2> \tag{6}$$

Invoking the Schwarz Inequality, it is clear from (5) and (6) that $\alpha_1 \beta$ is greater than 1, a proof pointed out to me by my student Chao Luo.

By considering the changes in the fluid kinetic energy resulting from a uniform volume change for every particle it can be shown [1] that the difference is mean pressure between the phases is

$$p_2 - p_1 = \frac{1}{2}\rho_1 w^2 \alpha_1 \frac{d}{d\alpha_1}(\alpha_1 \beta - 1) \tag{7}$$

The additional "1" in (7) appears to be gratuitous. It is introduced at this stage because the "exertia," defined [1] as

$$E = (\alpha_1 \beta - 1) \tag{8}$$

turns out to reappear in numerous contexts and to be one version of an *added mass coefficient.*

If we now superimpose a uniform velocity $\mathbf{v}_2$ on the above motion, this is equivalent to superimposing a corresponding potential gradient and we obtain

$$\mathbf{v}_1 + E\mathbf{w} = -\nabla\Phi \tag{9}$$

where the relative velocity is

$$\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2 \tag{10}$$

The net momentum and kinetic energy densities are then

$$\mathbf{m} = \rho_1 \alpha_1 \mathbf{v}_1 + \rho_2 \alpha_2 \mathbf{v}_2 \tag{11}$$

$$k = \frac{1}{2}\rho_1 \alpha_1 v_1^2 + \frac{1}{2}\rho_2 \alpha_2 v_2^2 + \frac{1}{2}\rho_1 \alpha_1 E w^2 \tag{12}$$

From (11) and (12) we may deduce that the effective *equations of motion* of a uniform suspension accelerating under the influence of a uniform macroscopic pressure gradient and body force fields are

$$\dot{\mathbf{v}}_1 + E\dot{\mathbf{w}} = -\frac{\nabla P}{\rho_1} + \mathbf{g}_1 + \frac{\mathbf{a}_1 \times \mathbf{w}}{\rho_1} \tag{13}$$

$$\dot{\mathbf{v}}_2 - \frac{\rho_1 \alpha_1}{\rho_2 \alpha_2} E\dot{\mathbf{w}} = -\frac{\nabla P}{\rho_2} + \mathbf{g}_2 + \frac{\mathbf{a}_2 \times \mathbf{w}}{\rho_2} \tag{14}$$

$\mathbf{a}_1$ and $\mathbf{a}_2$ are arbitrary vectors subject to the constraint

$$(\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2) \times \mathbf{w} = 0 \tag{15}$$

It would appear that the assembly must be incompressible if it is to be truly "uniform" in a pressure gradient. If $\mathbf{g}_1$ is a conservative force field, comparison between (9) and (13) would seem to indicate that $\nabla \times \mathbf{a}_1 \times \mathbf{w}$ is zero and therefore $\nabla \times \mathbf{a}_2 \times \mathbf{w}$ is as well from (15). Eq. (14) then suggests that if $\mathbf{g}_2$ is conservative, the combination on the left-hand side is the gradient of another "potential" which we could define in a form somewhat resembling (9):

$$\mathbf{v}_2 - \frac{\rho_1 \alpha_1}{\rho_2 \alpha_2} E\mathbf{w} = -\nabla \eta \tag{16}$$

A different argument is used to derive the one-dimensional equivalent of (16) in [1]. In each case, the interpretation of Φ and η could be, in the classical view, in terms of impulsive pressures and body forces necessary to set up the motion. This derivation, made for a uniform suspension, requires further argument, or a leap of faith, if it is to be applied more generally to a dispersion in which $\mathbf{w}$ and α_1, and hence β or E, vary with position.

Eqs. (13) and (14) show that the *exertia* plays the classical role of a *coefficient of apparent mass*, being proportional to various alternative definitions [1] in the literature.

Some other results derived from mechanistic arguments [1] are:

- The combined momentum flux and stress tensor for the suspension:

$$\mathbf{P} = \alpha_1(\rho_1\mathbf{v}_1\mathbf{v}_1 + p_1\mathbf{I}) + \alpha_2(\rho_2\mathbf{v}_2\mathbf{v}_2 + p_2\mathbf{I}) + \alpha_1\rho_1 E\mathbf{w}\mathbf{w} \tag{17}$$

- Bernoulli's equation for fluid flowing steadily past a stationary particle matrix:

$$p_1 = p_{01} - \frac{1}{2}\rho_1 w^2(1 + E) \tag{18}$$

- The force per unit volume on a particle in a stationary lattice:

$$\mathbf{f}_2 = \nabla p_2 = \nabla\left(\frac{1}{2}\rho_1 v_1^2 \alpha_1^2 \frac{d\beta}{d\alpha_1}\right) \tag{19}$$

When these results are used to check several hypothesized forms of the equations of motion in the *two-fluid model* of two-phase flow, there are found to be discrepancies [7], except when Geurst's equations are used.

GEURST'S EQUATIONS

In a series of papers [2-6] Geurst has used variational methods to derive a number of results, starting from the hypothesis that the kinetic energy per unit total volume is

$$k = \frac{1}{2}\rho_1\alpha_1 v_1^2 + \frac{1}{2}\rho_2\alpha_2 v_2^2 + \frac{1}{2}\rho_1 m w^2 \tag{20}$$

which is the same as (12) with the alternative definition

$$m = \alpha_1(\alpha_1\beta - 1) = \alpha_1 E \tag{21}$$

Initially Geurst [2,3] developed a one-dimensional theory, introducing Lagrange multipliers that are the same as the potentials Φ and η in (9) and (16). He went on to derive equations of motion, momentum, fluxes, pressures, etc. that are compatible with the results in Section 2, where there is a clear equivalence.

In three-dimensions the derivation is complicated by the introduction of Clebsch potentials and Lin constraints. Details are not provided in earlier publications [2-4] but they do appear in a recent one [6] in which the equivalents of (9) and (16) are expressed as

$$\boldsymbol{\pi}_1 = \nabla \phi_1 + \psi_1 \nabla \chi_1 \tag{22}$$

$$\boldsymbol{\pi}_2 = \nabla \phi_2 + \psi_2 \nabla \chi_2 + \frac{n}{\rho_2} \nabla \phi_n \tag{23}$$

where $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are *generalized specific momenta* defined as

$$\boldsymbol{\pi}_1 = \mathbf{v}_1 - \frac{m}{\alpha_1}(\mathbf{v}_2 - \mathbf{v}_1) \tag{24}$$

$$\boldsymbol{\pi}_2 = \mathbf{v}_2 + \frac{\rho_1 m}{\rho_2 \alpha_2}(\mathbf{v}_2 - \mathbf{v}_1) \tag{25}$$

which are identical with the left-hand sides of (9) and (16), in view of (21).

The ϕ's and χ's are five "potentials" or Lagrange multipliers and n is the bubble density. The detailed interpretation of these terms is not important in the present context.

Geurst's equations of motion appear in many equivalent forms. The versions selected in [8] may be written in the present notation as

$$\frac{\partial}{\partial t}(\rho_1 \alpha_1 \boldsymbol{\pi}_1) + \nabla \cdot (\rho_1 \alpha_1 \mathbf{v}_1 \boldsymbol{\pi}_1) = -\alpha_1 \nabla p_2 - \nabla \left[\frac{1}{2}\rho_1 (m + \alpha_1 m') w^2 \right]$$

$$+\alpha_1 \mathbf{f}_1 + \rho_1 m (\mathbf{v}_2 - \mathbf{v}_1) \cdot (\nabla \mathbf{v}_2)^T \qquad (26)$$

$$\frac{\partial}{\partial t}(\rho_2 \alpha_2 \boldsymbol{\pi}_2) + \nabla \cdot (\rho_2 \alpha_2 \mathbf{v}_2 \boldsymbol{\pi}_2) = -\alpha_2 \nabla p_2 + \alpha_2 \mathbf{f}_2 - \rho_1 m (\mathbf{v}_2 - \mathbf{v}_1) \cdot (\nabla \mathbf{v}_2)^T \qquad (27)$$

which have the form of conservation laws for $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$. m' denotes $dm/d\alpha_2$.

The two continuity equations are

$$\frac{\partial}{\partial t}(\rho_1 \alpha_1) + \nabla \cdot (\rho_1 \alpha_1 \mathbf{v}_1) = 0 \qquad (28)$$

$$\frac{\partial}{\partial t}(\rho_2 \alpha_2) + \nabla \cdot (\rho_2 \alpha_2 \mathbf{v}_2) = 0 \qquad (29)$$

If (7) is accepted and (21) is used, we find that the term involving w^2 on the right-hand side of (26) may be rewritten as

$$\frac{1}{2}\rho_1 (m + \alpha_1 m') w^2 = \alpha_1 (p_1 - p_2) \qquad (30)$$

Using (28) through (30) it is straightforward to arrange (26) and (27) into the forms

$$\frac{\partial}{\partial t}(\boldsymbol{\pi}_1) + \nabla \left(\mathbf{v}_1 \cdot \boldsymbol{\pi}_1 - \frac{v_1^2}{2} - \frac{1}{2}\frac{m}{\alpha_1}(\mathbf{v}_2 - \mathbf{v}_1)^2 \right) - \mathbf{v}_1 \times \nabla \times \boldsymbol{\pi}_1 = -\frac{\nabla p_1}{\rho_1} + \frac{\mathbf{f}_1}{\rho_1} \qquad (31)$$

$$\frac{\partial}{\partial t}(\boldsymbol{\pi}_2) + \nabla \left(\mathbf{v}_2 \cdot \boldsymbol{\pi}_2 - \frac{v_2^2}{2} \right) - \mathbf{v}_2 \times \nabla \times \boldsymbol{\pi}_2 = \frac{-\nabla p_2}{\rho_2} + \frac{\mathbf{f}_2}{\rho_2} \qquad (32)$$

Eqs. (31) and (32) are versions of the Bernoulli Equations derived in [8] without identifying $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ with gradients of potentials.

Now, if either ρ_1 is constant or ∇p_1 is parallel to $\nabla \rho_1$, and $\nabla \times (\mathbf{f}_1/\rho_1) = 0$, we may take the curl of (31) and obtain

$$\frac{\partial}{\partial t}(\nabla \times \boldsymbol{\pi}_1) - \nabla \times \mathbf{v}_1 \times (\nabla \times \boldsymbol{\pi}_1) = 0 \qquad (33)$$

which is a conservation law for the *vorticity* of $\boldsymbol{\pi}_1$. If we consider the rate of change of the net vorticity of $\boldsymbol{\pi}_1$ threading a loop moving with the velocity $\mathbf{v}_1$, we obtain

$$\frac{d}{dt}\int \boldsymbol{\pi}_1 \cdot \mathbf{d}\ell = \frac{d}{dt}\int (\nabla \times \boldsymbol{\pi}_1) \cdot \mathbf{d}s$$

$$= \int \frac{\partial}{\partial t}(\nabla \times \boldsymbol{\pi}_1) \cdot \mathbf{d}s + \int (\nabla \times \boldsymbol{\pi}_1) \cdot (\mathbf{v}_1 \times \mathbf{d}\ell) \qquad (34)$$

The two terms in (34) represent the sum of in-place changes in the flux through the loop and contributions picked up by the motion of boundaries. Changing the order of the scalar triple product in the final term and invoking Stokes' Theorem, we obtain a surface integral of (33) which is identically zero, in other words,

$$\frac{d}{dt}\int \boldsymbol{\pi}_1 \cdot \mathbf{d}\ell = 0 \qquad (35)$$

This result was obtained by Geurst [3]. A similar conclusion follows for $(\nabla \times \boldsymbol{\pi}_2)$.

Now, if both phases come from a region in which there is no curl to $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ (for example, a stagnation region), conservative body forces act and density gradients (if any) are parallel to pressure gradients, then throughout the flow, in view of (35), it should be true that

$$\nabla \times \boldsymbol{\pi}_1 = 0; \qquad \nabla \times \boldsymbol{\pi}_2 = 0 \qquad\qquad (36)$$

which implies that both $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are gradients of suitable potentials and only the first term is needed on the right-hand sides of (22) and (23) which reduce to the form of (9) and (16). This development apparently makes more explicit the conditions for the existence of *two-phase potential flow*, requirements which parallel those for the classical single-phase case.

DISCUSSION

The previous sections of this paper have outlined two approaches to *potential two-phase flow*. Each has a justifiable theoretical base and is self-consistent. Moreover, both approaches appear to give the "right" prediction for several well-defined situations [7] while some other formulations fail these tests.

In order for these ideas to blossom further there need to be:

a) Further developments, from the same basic set of assumptions, that encompass more generality.

b) More rigorous derivations that clearly explain the order of approximation involved in treating the flow of discrete entities as a continuum.

c) Reconciliation with alternative approaches, particularly those involving *averaging*.

d) More solutions to specific problems that can be thoroughly investigated for consistency.

e) An understanding of outstanding incompatibilities between these approaches and various other theories, with a clear explanation of what has "gone wrong."

I hope that some of these items can be discussed at this meeting. At this time I only wish to briefly address (d) and (e).

Apart from the "tests" described in [1] and [7] and perturbation techniques leading to the description of wave propagation and stability in bubbly flow [2,3,5], I know of only two complete solutions to Geurst's equations. The first involves steady flow of two incompressible fluids from a common stagnation region. The simple conclusion that is reached is that the void fraction is constant and the velocities of both phases are the gradients of potentials that are proportional to each other and may be borrowed from an equivalent single-phase flow. The velocity ratio, or *slip ratio* is

$$\frac{\mathbf{v}_2}{\mathbf{v}_1} = \left(\frac{3R}{R+2} \right)^{1/2} \tag{37}$$

where

$$R = \rho_1 / \rho_2 \tag{38}$$

In the above derivation use was made of Maxwell's [9] approximate expression for the *exertia*,

$$E \approx \frac{\alpha_2}{2} \tag{39}$$

The second solution again uses (39) and leads to a relationship between the velocities in unsteady incompressible flow [8]

$$\left(\rho_2 + \frac{\rho_1}{2} \right) \left(\frac{\partial \mathbf{v}_2}{\partial t} + \frac{\nabla v_2^2}{2} \right) - \frac{3}{2} \rho_1 \left(\frac{\partial \mathbf{v}_1}{\partial t} + \frac{\nabla v_1^2}{2} \right) + \mathbf{f}_1 - \mathbf{f}_2 = 0 \tag{40}$$

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This has the same form as a relationship which would be obtained by eliminating the pressures from equations of motion that ignored inertial coupling, except that the effective densities are changed. There is opportunity to test (40) by comparison with the one-dimensional transient response of fluidized beds.

Regarding item (e) I will repeat, in a slightly different form, Geurst's equations of motion [4] as presented in [8]

$$\frac{\partial}{\partial t}(\rho_1 \alpha_1 \mathbf{v}_1) + \nabla \cdot (\rho_1 \alpha_1 \mathbf{v}_1 \mathbf{v}_1 + \rho_1 m \mathbf{w} \mathbf{w}) =$$

$$-\alpha_1 \nabla p_1 - (p_1 - p_2)\nabla \alpha_1 + M_1^d + \alpha_1 \mathbf{f}_1 \tag{41}$$

$$\frac{\partial}{\partial t}(\rho_2 \alpha_2 \mathbf{v}_2) + \nabla \cdot (\rho_2 \alpha_2 \mathbf{v}_2 \mathbf{v}_2) = -\alpha_2 \nabla p_2 - M_1^d + \alpha_2 \mathbf{f}_2 \tag{42}$$

with the added mass term expressed as

$$M_1^d = \rho_1 m \left[\left(\frac{\partial}{\partial t} + \mathbf{v}_2 \cdot \nabla \right) \mathbf{v}_2 - \left(\frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \nabla \right) \mathbf{v}_1 + \mathbf{w} \times \nabla \times \mathbf{v}_1 \right]$$

$$+ \rho_1 m \nabla \frac{w^2}{2} + \mathbf{w} \left[\frac{\partial}{\partial t}(\rho_1 m) + \nabla \cdot \rho_1 m \mathbf{v}_2 \right] \tag{43}$$

This set of equations contains several more terms than one would find in most similar expressions in the literature. By dint of these extra terms, several "tests" are passed that other formulations fail [7]. Specifically, these tests involve (18) and (19), which reduces for a dilute dispersion to Taylor's expression [10] for the force on a stationary object in an accelerating flow. It would be desirable to devise other "tests" that might help to discriminate further between true and false expressions.

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